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On the Derivation of a Sixth-Stage-Fifth-Order Runge–Kutta Method for Solving Initial Value Problems in Ordinary Differential Equations

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ABSTRACT: *In this paper, a new Sixth-Stage Fifth-Order Runge-Kutta Formula was derived and implemented. The results obtained, were compared with existing methods in literature, in other to determine the level of performance of the method. The consistency and convergence properties of the method were properly investigated. Errors involved in the new method and that of kutta-Nystrom method, were plotted with MATLAB to obtain their trajectories.*

Keywords: *Initial value problems, Taylor series, Consistency, Convergence, and Error Analysis*

I. INTRODUCTION

The solutions of some real life problems can be modeled through the process of differential equations. Such differential equations may not be solved by the use of analytical techniques. If the differential equation is not solvable by analytical, then we can resort to numerical methods. Many methods have been proposed and used by different authors in an attempt to provide accurate solutions to the various types of differential equations arising from the models, Bazuaye [1]. The focus here is to propose a technique that can solve problems in ordinary differential equations which arise frequently in several models of mathematical physics, biological Sciences, engineering and applied mathematics.

There are various classes of methods in this direction, such as Taylor series method, Euler method, Block method, multi- linear -step method, Runge-Kutta method and many other methods. However, this work will be on special types of explicit Runge-Kutta method aimed at providing solutions to Initial Value Problems (IVPs) in ordinary differential equations. Although, various formulae have been developed from this general Runge-kutta method, such as the popular fourth -order Runge-Kutta , fifth- order Runge-Kutta, a cubic Root mean fourth-order Runge-Kutta, Geometric mean fourth-order Runge-Kutta method etc, Aashikpelokhai and Agbeboh [3], Agbeboh and Aashikpelokhai [4]. According to Abbas [2], the 4^{th} order explicit Runge-Kutta method has become the most popular version of the classical Runge-Kutta method. The fourth-order Runge-Kutta method is now recognized as the starting point for the modern one-step methods.Butcher [5] noted that "Runge-Kutta method provide a suitable way of numerical solutions to ordinary differential equations". Butcher [5] further revealed that, "Different approaches have been used by many authors in the past to derive a method that will minimize the error associated with the use of Runge-Kutta method".

Agbeboh and Esekhaigbe [6] also noted that "…….despite the evolutions of a vast and comprehensive body of knowledge, Explicit Runge-Kutta algorithms continue to be sources of active research". He further noted that Runge_-Kutta methods represent an important family of explicit and implicit iterative methods of approximation of solution to ordinary differential equations in numerical analysis.

According to Islam [7], the Runge-Kutta method is most popular because it is quite accurate, stable and easy to program. This method is distinguished by their order in the sense that they agree with Taylor's series solution up to terms of h^r, where r is the order of the method. It does not demand prior computation of higher derivatives of $y(x)$ as in Taylor's series method. Agbeboh and Omonkaro in [8] also noted that, the philosophy behind the Runge-Kutta methods, is to retain the advantages of one-step methods and to improve on the performance of Euler method. He further said due to loss of linearity in the one-step methods, error analysis is considerably more difficult than the case of multi- linear step method. Traditionally, Runge-Kutta methods are all explicit method, although, implicit Runge-Kutta methods have extensively been used to improved weak stability characteristics.

According to Agbeboh and Ehiemua [9], construction of Runge-Kutta methods need the partial derivative of the functions whose parameters are sort by comparing the emerging equations with Taylor series expansion, because they involve so many non-linear algebraic expressions called the "order" conditions. They further noted that, there is a tedious manipulation in the process of deriving Runge-Kutta methods of higher order. He noted that, in deriving a fourth order Runge –Kutta method, a system of 11 equations in 13 unknowns are obtained as against 4 equations in 6 unknown, in the case of third order methods. This lack of uniqueness is typical of all Runge-Kutta methods.

Many researchers have done a great deal of work in the area of getting a more suitable method that is efficient in handling singular initial value problems in ordinary differential equations with minimum error bound. Worthy of note are those of Agbeboh and Esekhaigbe [6], Agbeboh and Ehiemua [9], Lambert [10], Butcher [5] and host of others. In their various efforts, they believe that if parameters are carefully varied, the tendency is to have method that will possess the potentials of improving results and thereby reducing error.Furthermore, Lambert [10] acknowledge that there are many various existing methods for the solutions of initial value problem in ordinary differential equation, but not all such methods have the capacity of providing results with high accuracy to these initial value problems. For that reason, we were motivated to derive Sixth-Stage Fifth-Order Runge-kutta method that will provide solutions with higher accuracy and lower level of error to these initial value problems. However, the derivation of fifth-order Runge-Kutta method was introduced by Kutta and advanced by Nystrom as stated in Abbas [2]. Therefore, the overall aim of this paper is to developing a numerical method for the solution of the initial value problem of the type: $y' = f(x,y)$, $y(x_0) = y_0, x \in [a,b]$ (1)

Where gradient function f(x, y) may have points of discontinuities,

While the specific objectives are to:

(a) deriving a Sixth-Stage Fifth-Order Runge-Kutta method with high accuracy with a minimal error for solving initial value problem;

(b) determine the consistence and convergence nature of the method;

(c) implement and compared the performance of the new method with Kutta-Nystrom sixth-stage fifth-order method using some tested initial value problems. A display of solution tables will be provided as a way of comparing both numerical results. For the purpose of clarity the following definitions are necessary.

Consider the general one-step explicit Runge-Kutta method given by;

$$y_{n+1} - y_n = h\phi(x_n, y_n, h) \quad (2)$$

Lambert [12] defining

The general one-step method (2) is said to be of order P, if P is the largest integer for which

$$y(x+h) - y(x) - h\phi(x, y(x), h) = o\left(h^{p+1}\right) \quad (3)$$

holds, where $y(x)$ is the theoretical solution of the initial value problem

Lambert [12], also define the general one-step method (2) be consistent with the initial problem (1) if:

$$\phi(x, y, 0) \equiv f(x, y) \quad (4)$$

Then we can say that the method in equation (2) is consistent with the initial value problem, furthermore, if:

$$y(x+h)-y(x)-h\phi(x,y(x),h)=hy'(x)-h\phi(x,y(x),0)+o(h^2) \quad (5)$$

Since $y'(x)=f(x,y(x))=\phi(x,y(x),0)$ by equation (4), thus a consistent method has order of at least one.

Another important definition given in Lambert [12] is that, the local truncation error at x_{n+1} of the general explicit one-step method in (2) defined as T_{n+1}, where

$$T_{n+1}=y(x_{n+1})-y(x)-h\phi(x,y(x),h) \quad (6)$$

And $y(x)$ is the theoretical solution of the initial value problem

Furthermore, the general one-step method (2) is said to be convergent if the initial value problem in (1), has the corresponding approximation y_n satisfying $\text{limit}(y_n)\rightarrow y(x)$ as $n\rightarrow\infty$ (7)

II. Derivation of the Method

The sixth-stage fifth-order Runge-Kutta formula can be obtained using the following procedure:
From the general Runge-Kutta method, get a sixth-stage fifth-order method;
Obtain the Taylor series expansion of $k_{i's}$ about the point (y_n), $i=1,2,3,4,5,6$
Carry out substitution to ensured that all $k_{i's}$ are in terms of k_1 only
Reducing all the $k_{i's}$ in terms of k_1 and substituting into the increment function;. $\phi(y_n,h)=\sum_{i=1}^{6}b_i k_i$,
followed by some simplification we get the required Runge-Kutta method by comparing the coefficients of all partial derivatives of y with Taylor series expansion of fifth-order involving only partial derivatives with respect to y as shown in the derivation below.
As a result, a set of 12 linear and non linear equations will be generated by varying the parameters of the sets of equations generated. After all that, a new sixth-stage fifth-order explicit Runge-Kutta formula will be obtained as follows:
From equation (2), the general Runge-Kutta method is defined as:

$$y_{n+1}-y_n=h\phi(x_n,y_n,h)$$

$$\phi(x_n,y_n,h)=\sum_{i=1}^{R}b_i k_i \qquad i=1,2,3,\ldots 6 \quad (8)$$

$$k_1=f(x_n,y_n) \quad =f \quad (9)$$

$$k_i=f(x_n+hc_i,y_n+h\sum_{j=1}^{i-1}a_{ij}k_j) \qquad i=2,3,\ldots 6 \quad (10)$$

$$c_i=\sum_{j=1}^{i-1}a_{ij} \quad , \quad i=2,3\ldots 6 \quad (11)$$

Since we are looking for a method with six stages, then we shall adopt the Taylor series to expand (10), by setting i = 1, 2,3,4,5 and 6 as;

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + hc_2, y_n + h(a_{21}k_1))$$
$$k_3 = f(x_n + hc_3, y_n + h(a_{31}k_1 + a_{32}k_2))$$
$$k_4 = f(x_n + hc_4, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3))$$

$$k_6 = f(x_n + hc_6, y_n + h(a_{61}k_1 + a_{62}k_2 + a_{63}k_3 + a_{64}k_4 + a_{65}k_5))$$

For the purpose of linearity, the above parameters will be modified as follows:

$a_{21} = a_1$, $a_{31} = a_2$, $a_{32} = a_3$, $a_{41} = a_4$, $a_{42} = a_5$,
$a_{43} = a_6$, $a_{51} = a_7$, $a_{52} = a_8$, $a_{53} = a_9$, $a_{54} = a_{10}$,
$a_{61} = a_{11}$, $a_{62} = a_{12}$, $a_{63} = a_{13}$, $a_{64} = a_{14}$, $a_{65} = a_{15}$

Substituting we have;

$$k_1 = f(y_n) \qquad (12)$$

$$k_2 = f(y_n + h(a_1k_1)) \qquad (13)$$

$$k_3 = f(y_n + h(a_2k_1 + a_3k_2)) \qquad (14)$$

$$k_4 = f(y_n + h(a_4k_1 + a_5k_2 + a_6k_3)) \qquad (15)$$

$$k_5 = f(y_n + h(a_7k_1 + a_8k_2 + a_9k_3 + a_{10}k_4)) \qquad (16)$$

$$k_6 = f(y_n + h(a_{11}k_1 + a_{12}k_2 + a_{13}k_3 + a_{14}k_4 + a_{15}k_5)) \qquad (17)$$

Adopting Taylor series expansion about the point (y_n), i.e., discarding all the derivatives of x and leaving those of y alone, we have:

$$k_1 = f(y_n) \qquad (18a)$$

$$k_2 = \sum_{i=0}^{\infty} \frac{1}{i!}\left(y + ha_1k_1 \frac{d}{dy} \right)^i f(y_n) \qquad (18b)$$

$$k_3 = \sum_{i=0}^{\infty} \frac{1}{i!}\left(y + h\left(a_2k_1 + a_3k_2\right) \frac{d}{dy} \right)^i f(y_n) \qquad (18c)$$

$$k_4 = \sum_{i=0}^{\infty} \frac{1}{i!}\left(y + h(a_4k_1 + a_5k_2 + a_6k_3) \frac{d}{dy} \right)^i f(y_n) \qquad (18d)$$

$$k_5 = \sum_{i=0}^{\infty} \frac{1}{i!}\left(y + h(a_7k_1 + a_8k_2 + a_9k_3 + a_{10}k_4) \frac{d}{dy} \right)^i f(y_n) \qquad (18e)$$

$$k_6 = \sum_{i=0}^{\infty} \frac{1}{i!}\left(y + h(a_{11}k_1 + a_{12}k_2 + a_{13}k_3 + a_{14}k_4 + a_{15}k_5) \frac{d}{dy} \right)^i f(y_n) \qquad (18f)$$

Expanding (18a) to (18f) for $k_1, \ldots, k_6$, we have the following;

$k_1 = f$ (19)

$$k_2 = k_1 + h\,a_1\,k_1 f_y + \frac{1}{2}\,h^2\,a_1^2\,k_1^2 f_{yy} + \frac{1}{6}\,h^3\,a_1^3\,k_1^3 f_{yyy} + \frac{1}{24}\,h^4\,a_1^4\,k_1^4 f_{yyyy} \tag{20}$$

$$k_3 = k_1 + k_1\,c_3 f_y\,h + \frac{1}{2}\,h^2\,k_1^2\,c_3^2 f_{yy} + h^2\,a_3\,a_1\,k_1 f_y^2 + \frac{1}{2}\,h^3\,a_3\,a_1\,k_1^2 f_y\left(a_1 + 2\,c_3\right) f_{yy} + \frac{1}{6}\,h^3$$
$$k_1^3\,c_3^3 f_{yyy} + \frac{1}{2}\,h^4\,a_1^2\,a_3\,k_1^3\,c_3 f_{yy}^2 + \frac{1}{2}\,h^4\,a_1^2\,a_3^2 f_y^2 f_{yy} k_1^2 + \frac{1}{6}\,h^4\,a_3\,a_1\,k_1^3 f_y\left(a_1^2 + 3\,c_3^2\right) f_{yyy}$$
$$+ \frac{1}{24}\,h^4\,k_1^4\,c_3^4 f_{yyyy}$$

(21)

$$k_4 = k_1 + k_1\,c_4 f_y\,h + h^2\,k_1\left(a_1\,a_5 + a_6\,c_3\right) f_y^2 + \frac{1}{2}\,h^2\,k_1^2\,c_4^2 f_{yy} + h^3\,a_6\,a_3\,a_1\,k_1 f_y^3 + \frac{1}{2}\,h^3\,k_1^2\,\Big($$
$$c_3^2\,a_6 + 2\,c_3\,a_6\,c_4 + a_1\,a_5\left(2\,c_4 + a_1\right)\Big) f_{yy} f_y + \frac{1}{6}\,h^3\,k_1^3\,c_4^3 f_{yyy} + \frac{1}{2}\,h^4 f_y^2\,k_1^2\Big(c_3^2\,a_6^2$$
$$+ 2\,c_3\,a_1\,a_6\left(a_3 + a_5\right) + a_1^2\,a_3\,a_6 + a_1^2\,a_5^2 + 2\,a_1\,a_3\,a_6\,c_4\Big) f_{yy} + \frac{1}{6}\,h^4\,k_1^3\Big(a_1^3\,a_5 + 3\,a_1\,a_5\,c_4^2$$
$$+ a_6\,c_3^3 + 3\,a_6\,c_3\,c_4^2\Big) f_{yyy} f_y + \frac{1}{2}\,h^4\,k_1^3\left(a_1^2\,a_5 + a_6\,c_3^2\right) c_4 f_{yy}^2 + \frac{1}{24}\,h^4\,k_1^4\,c_4^4 f_{yyyy}$$

(22)

$$k_5 = k_1 + k_1\,c_5 f_y\,h + h^2\,k_1\left(a_1\,a_8 + a_9\,c_3 + a_{10}\,c_4\right) f_y^2 + \frac{1}{2}\,h^2\,k_1^2\,c_5^2 f_{yy} + h^3\,k_1\,\Big(a_1\,a_3\,a_9$$
$$+ a_1\,a_5\,a_{10} + a_6\,a_{10}\,c_3\Big) f_y^3 + \frac{1}{2}\,h^3\,k_1^2\Big(a_1^2\,a_8 + 2\,a_1\,a_8\,c_5 + a_9\,c_3^2 + 2\,a_9\,c_3\,c_5 + a_{10}\,c_4^2$$
$$+ 2\,a_{10}\,c_4\,c_5\Big) f_{yy} f_y + \frac{1}{6}\,h^3\,k_1^3\,c_5^3 f_{yyy} + h^4\,a_{10}\,a_6\,a_3\,a_1\,k_1 f_y^4 + \frac{1}{2}\,h^4\,k_1^2\Big(c_3^2\,a_6\,a_{10} + c_3^2\,a_9^2$$
$$+ 2\,c_3\,c_4\,a_{10}\left(a_6 + a_9\right) + a_1\,a_3\,a_9\left(a_1 + 2\,c_3\right) + 2\,c_3\,a_1\,a_8\,a_9 + 2\,c_3\,a_6\,a_{10}\,c_5 + c_4^2\,a_{10}^2$$
$$+ a_1\,a_{10}\left(a_1\,a_5 + 2\,a_5\,c_4 + 2\,a_8\,c_4\right) + a_1^2\,a_8^2 + 2\,a_1\,a_3\,a_9\,c_5 + 2\,a_1\,a_5\,a_{10}\,c_5\Big) f_{yy} f_y^2 + \frac{1}{6}\,h^4$$
$$k_1^3\left(a_1^3\,a_8 + 3\,a_1\,a_8\,c_5^2 + a_9\,c_3^3 + 3\,a_9\,c_3\,c_5^2 + a_{10}\,c_4^3 + 3\,a_{10}\,c_4\,c_5^2\right) f_{yyy} f_y + \frac{1}{2}\,h^4\,k_1^3\Big(a_1^2\,a_8$$
$$+ a_9\,c_3^2 + a_{10}\,c_4^2\Big)\,c_5 f_{yy}^2 + \frac{1}{24}\,h^4\,k_1^4\,c_5^4 f_{yyyy}$$

(23)

$$k_6 = k_1 + k_1\,c_6 f_y\,h + h^2\,k_1\left(a_1\,a_{12} + a_{13}\,c_3 + a_{14}\,c_4 + a_{15}\,c_5\right) f_y^2 + \frac{1}{2}\,h^2\,k_1^2\,c_6^2 f_{yy}$$
$$+ h^3\,k_1\left(a_1\,a_3\,a_{13} + a_1\,a_5\,a_{14} + a_1\,a_8\,a_{15} + a_6\,a_{14}\,c_3 + a_9\,a_{15}\,c_3 + a_{10}\,a_{15}\,c_4\right) f_y^3 + \frac{1}{2}\,h^3$$
$$k_1^2\Big(a_1^2\,a_{12} + 2\,a_1\,a_{12}\,c_6 + a_{13}\,c_3^2 + 2\,a_{13}\,c_3\,c_6 + a_{14}\,c_4^2 + 2\,a_{14}\,c_4\,c_6 + a_{15}\,c_5^2$$
$$+ 2\,a_{15}\,c_5\,c_6\Big) f_{yy} f_y + \frac{1}{6}\,h^3\,k_1^3\,c_6^3 f_{yyy} + h^4\,k_1\,\Big(c_3\,a_6\,a_{10}\,a_{15} + a_1\,a_3\left(a_6\,a_{14} + a_9\,a_{15}\right)$$
$$+ a_1\,a_5\,a_{10}\,a_{15}\Big) f_y^4 + \frac{1}{2}\,h^4\,k_1^2\Big(a_1^2\,a_8\,a_{15} + a_1^2\,a_5\,a_{14} + c_3^2\,a_6\,a_{14} + c_3^2\,a_9\,a_{15} + c_4^2\,a_{10}\,a_{15}$$
$$+ c_3^2\,a_{13}^2 + c_4^2\,a_{14}^2 + c_5^2\,a_{15}^2 + a_1^2\,a_{12}^2 + 2\,c_3\,c_5\,a_{13}\,a_{15} + a_1\,a_3\,a_{13}\left(a_1 + 2\,c_3\right)$$
$$+ 2\,c_3\,a_1\,a_{12}\,a_{13} + 2\,c_3\,a_6\,a_{14}\,c_6 + 2\,c_3\,a_9\,a_{15}\,c_6 + 2\,c_4\,c_5\,a_{10}\,a_{15} + 2\,c_4\,c_5\,a_{14}\,a_{15}$$
$$+ 2\,c_4\,a_1\,a_5\,a_{14} + 2\,c_4\,a_1\,a_{12}\,a_{14} + 2\,c_4\,a_{10}\,a_{15}\,c_6 + 2\,c_5\,a_1\,a_8\,a_{15} + 2\,c_5\,a_1\,a_{12}\,a_{15}$$
$$+ 2\,a_1\,a_3\,a_{13}\,c_6 + 2\,a_1\,a_5\,a_{14}\,c_6 + 2\,a_1\,a_8\,a_{15}\,c_6 + 2\,c_3\,c_4\,a_6\,a_{14} + 2\,c_3\,c_4\,a_{13}\,a_{14}$$
$$+ 2\,c_3\,c_5\,a_9\,a_{15}\Big) f_{yy} f_y^2 + \frac{1}{6}\,h^4\,k_1^3\Big(a_1^3\,a_{12} + 3\,a_1\,a_{12}\,c_6^2 + a_{13}\,c_3^3 + 3\,a_{13}\,c_3\,c_6^2 + a_{14}\,c_4^3$$
$$+ 3\,a_{14}\,c_4\,c_6^2 + a_{15}\,c_5^3 + 3\,a_{15}\,c_5\,c_6^2\Big) f_{yyy} f_y + \frac{1}{2}\,h^4\,k_1^3\Big(a_1^2\,a_{12} + a_{13}\,c_3^2 + a_{14}\,c_4^2 + a_{15}$$
$$c_5^2\Big)\,c_6 f_{yy}^2 + \frac{1}{24}\,h^4\,k_1^4\,c_6^4 f_{yyyy}$$

(24)

Substituting $k_{i's}$ into $y_{n+1} - y_n = h\phi(x_n, y_n, h)$

Where $h\phi(x_n, y_n, h) = \sum_{i=1}^{R} b_i k_i$ and simplifying we have equation (25)

$$y_{n+1} - y_n = hk_1\left(b_1 + b_2 + b_3 + b_4 + b_5 + b_6\right) + h^2 k_1\left(a_1 b_2 + b_3 c_3 + b_4 c_4 + b_5 c_5 + b_6 c_6\right) f_y + h^3 k_1\left(a_1 a_3 b_3 + a_1 a_5 b_4 + a_1 a_8 b_5 + a_1 a_{12} b_6 + a_6 b_4 c_3 + a_9 b_5 c_3 + a_{10} b_5 c_4 + a_{13} b_6 c_3 + a_{14} b_6 c_4 + a_{15} b_6 c_5\right) f_y^2 + \frac{1}{2} h^3 k_1^2\left(a_1^2 b_2 + b_3 c_3^2 + b_4 c_4^2 + b_5 c_5^2 + b_6 c_6^2\right) f_{yy} + h^4 k_1\left(a_1 a_3 a_6 b_4 + a_1 a_3 a_9 b_5 + a_1 a_3 a_{13} b_6 + a_1 a_5 a_{10} b_5 + a_1 a_5 a_{14} b_6 + a_1 a_8 a_{15} b_6 + a_6 a_{10} b_5 c_3 + a_6 a_{14} b_6 c_3 + a_9 a_{15} b_6 c_3 + a_{10} a_{15} b_6 c_4\right) f_y^3 + \frac{1}{2} h^4 k_1^2\left(a_1^2 a_3 b_3 + a_1^2 a_5 b_4 + a_1^2 a_8 b_5 + a_1^2 a_{12} b_6 + 2 a_1 a_3 b_3 c_3 + 2 a_1 a_5 b_4 c_4 + 2 a_1 a_8 b_5 c_5 + 2 a_1 a_{12} b_6 c_6 + a_6 b_4 c_3^2 + 2 a_6 b_4 c_3 c_4 + a_9 b_5 c_3^2 + 2 a_9 b_5 c_3 c_5 + a_{10} b_5 c_4^2 + 2 a_{10} b_5 c_4 c_5 + a_{13} b_6 c_3^2 + 2 a_{13} b_6 c_3 c_6 + a_{14} b_6 c_4^2 + 2 a_{14} b_6 c_4 c_6 + a_{15} b_6 c_5^2 + 2 a_{15} b_6 c_5 c_6\right) f_{yy} f_y + \frac{1}{6} h^4 k_1^3\left(a_1^3 b_2 + b_3 c_3^3 + b_4 c_4^3 + b_5 c_5^3 + b_6 c_6^3\right) f_{yyy} + h^5 k_1\left(a_1 a_3 a_6 a_{10} b_5 + a_1 a_3 a_6 a_{14} b_6 + a_1 a_3 a_9 a_{15} b_6 + a_1 a_5 a_{10} a_{15} b_6 + a_6 a_{10} a_{15} b_6 c_3\right) f_y^4 + \frac{1}{2} h^5 k_1^2\left(a_1^2 a_3^2 b_3 + a_1^2 a_3 a_6 b_4 + a_1^2 a_3 a_9 b_5 + a_1^2 a_3 a_{13} b_6 + a_1^2 a_5^2 b_4 + a_1^2 a_5 a_{10} b_5 + a_1^2 a_5 a_{14} b_6 + a_1^2 a_8^2 b_5 + a_1^2 a_8 a_{15} b_6 + a_1^2 a_{12}^2 b_6 + 2 a_1 a_3 a_6 b_4 c_3 + 2 a_1 a_3 a_6 b_4 c_4 + 2 a_1 a_3 a_9 b_5 c_3 + 2 a_1 a_3 a_9 b_5 c_5 + 2 a_1 a_3 a_{13} b_6 c_3 + 2 a_1 a_3 a_{13} b_6 c_6 + 2 a_1 a_5 a_6 b_4 c_3 + 2 a_1 a_5 a_{10} b_5 c_4 + 2 a_1 a_5 a_{10} b_5 c_5 + 2 a_1 a_5 a_{14} b_6 c_4 + 2 a_1 a_5 a_{14} b_6 c_6 + 2 a_1 a_8 a_9 b_5 c_3 + 2 a_1 a_8 a_{10} b_5 c_4 + 2 a_1 a_8 a_{15} b_6 c_5 + 2 a_1 a_8 a_{15} b_6 c_6 + 2 a_1 a_{12} a_{13} b_6 c_3 + 2 a_1 a_{12} a_{14} b_6 c_4 + 2 a_1 a_{12} a_{15} b_6 c_5 + a_6^2 b_4 c_3^2 + a_6 a_{10} b_5 c_3^2 + 2 a_6 a_{10} b_5 c_3 c_4 + 2 a_6 a_{10} b_5 c_3 c_5 + a_6 a_{14} b_6 c_3^2 + 2 a_6 a_{14} b_6 c_3 c_4 + 2 a_6 a_{14} b_6 c_3 c_6 + a_9^2 b_5 c_3^2 + 2 a_9 a_{10} b_5 c_3 c_4 + a_9 a_{15} b_6 c_3^2 + 2 a_9 a_{15} b_6 c_3 c_5 + 2 a_9 a_{15} b_6 c_3 c_6 + a_{10}^2 b_5 c_4^2 + a_{10} a_{15} b_6 c_4^2 + 2 a_{10} a_{15} b_6 c_4 c_5 + 2 a_{10} a_{15} b_6 c_4 c_6 + a_{13}^2 b_6 c_3^2 + 2 a_{13} a_{14} b_6 c_3 c_4 + 2 a_{13} a_{15} b_6 c_3 c_5 + a_{14}^2 b_6 c_4^2 + 2 a_{14} a_{15} b_6 c_4 c_5 + a_{15}^2 b_6 c_5^2\right) f_{yy} f_y^2 + \frac{1}{6} h^5 k_1^3\left(a_1^3 a_3 b_3 + a_1^3 a_5 b_4 + a_1^3 a_8 b_5 + a_1^3 a_{12} b_6 + 3 a_1 a_3 b_3 c_3^2 + 3 a_1 a_5 b_4 c_4^2 + 3 a_1 a_8 b_5 c_5^2 + 3 a_1 a_{12} b_6 c_6^2 + a_6 b_4 c_3^3 + 3 a_6 b_4 c_3 c_4^2 + a_9 b_5 c_3^3 + 3 a_9 b_5 c_3 c_5^2 + a_{10} b_5 c_4^3 + 3 a_{10} b_5 c_4 c_5^2 + a_{13} b_6 c_3^3 + 3 a_{13} b_6 c_3 c_6^2 + a_{14} b_6 c_4^3 + 3 a_{14} b_6 c_4 c_6^2 + a_{15} b_6 c_5^3 + 3 a_{15} b_6 c_5 c_6^2\right) f_{yyy} f_y + \frac{1}{2} h^5 k_1^3\left(a_1^2 a_3 b_3 c_3 + a_1^2 a_5 b_4 c_4 + a_1^2 a_8 b_5 c_5 + a_1^2 a_{12} b_6 c_6 + a_6 b_4 c_3^2 c_4 + a_9 b_5 c_3^2 c_5 + a_{10} b_5 c_4^2 c_5 + a_{13} b_6 c_3^2 c_6 + a_{14} b_6 c_4^2 c_6 + a_{15} b_6 c_5^2 c_6\right) f_{yy}^2 + \frac{1}{24} h^5 k_1^4\left(a_1^4 b_2 + b_3 c_3^4 + b_4 c_4^4 + b_5 c_5^4 + b_6 c_6^4\right) f_{yyyy} \quad (25)$$

Finally, we compared equation (25) with Taylor series expansion of order five given thus;

$$y_{n+1} - y_n = hk_1 + \frac{1}{2}h^2 k_1 f_y + \frac{1}{6}h^3(k_1 f_y^2 + k_1^2 f_{yy}) + \frac{1}{24}h^4(k_1 f_y^3 + 4k_1^2 f_y f_{yy} + k_1^3 f_{yyy}) + \frac{1}{120}h^5(k_1 f_y^4 + 11k_1^2 f_{yy} f_y^2 + 7k_1^3 f_y f_{yyy} + 4k_1^3 f_{yy}^2 + k_1^4 f_{yyyy})$$

to arrive at the following set of parametric equations:

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 1 \quad (26a)$$

$$a_1 b_2 + b_3 c_3 + b_4 c_4 + b_5 c_5 + b_6 c_6 = \frac{1}{2} \quad (26b)$$

$$a_1 a_3 b_3 + a_1 a_5 b_4 + a_1 a_8 b_5 + a_1 a_{12} b_6 + a_6 b_4 c_3 + a_9 b_5 c_3 + a_{10} b_5 c_4 + a_{13} b_6 c_3 + a_{14} b_6 c_4 + a_{15} b_6 c_5 = \frac{1}{6} \quad (26c)$$

$$a_1^2 b_2 + b_3 c_3^2 + b_4 c_4^2 + b_5 c_5^2 + b_6 c_6^2 = \frac{1}{3} \quad (26d)$$

$$a_1a_3a_6b_4 + a_1a_3a_9b_5 + a_1a_3a_{13}b_6 + a_1a_5a_{10}b_5 + a_1a_5a_{14}b_6 + a_1a_8a_{15}b_6 + a_6a_{10}b_5c_3 + a_6a_{14}b_6c_3 + a_9a_{15}b_6c_3 + a_{10}a_{15}b_6c_4 = \frac{1}{24}$$

(26e)

$$a_1^2a_3b_3 + a_1^2a1_5b_4 + a_1^2a_8b_5 + a_1^2a_{12}b_6 + 2a_1a_3b_3c_3 + 2a_1a_5b_4c_4 + 2a_1a_8b_5c_5 + 2a_1a_{12}b_6c_6 + a_6b_4c_3^2 + 2a_6b_4c_3c_4 + a_9b_5c_3^2 + 2a_9b_5c_3c_5 + a_{10}b_5c_4^2 + 2a_{10}b_5c_4c_5 + a_{13}b_6c_3^2 + 2a_{13}b_6c_3c_6 + a_{14}b_6c_4^2 + 2a_{14}b_6c_4c_6 + a_{15}b_6c_5^2 + 2a_{15}b_6c_5c_6 = \frac{1}{3}$$

(26f)

$$a_1^3b_2 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 + b_6c_6^3 = \frac{1}{4}$$

(26g)

$$a_1a_3a_6a_{10}b_5 + a_1a_3a_6a_{14}b_6 + a_1a_3a_9a_{15}b_6 + a_1a_5a_{10}a_{15}b_6 + a_6a_{10}a_{15}b_6c_3 = \frac{1}{120}$$

(26h)

$$a_1^2a_3^2b_3 + a_1^2a_3a_6b_4 + a_1^2a_3a_9b_5 + a_1^2a_3a_{13}b_6 + a_1^2a_5^2b_5 + a_1^2a_5a_{10}b_5 + a_1^2a_5a_{14}b_6 + a_1^2a_8^2b_5 + a_1^2a_8a_{15}b_6 + a_1^2a_{12}^2b_6 + 2a_1a_3a_6b_4c_3 + 2a_1a_3a_6b_4c_4 + 2a_1a_3a_9b_5c_3 + 2a_1a_3a_9b_5c_5 + 2a_1a_3a_{13}b_6c_3 + 2a_1a_3a_{13}b_6c_6 + 2a_1a_5a_6b_4c_3 + 2a_1a_5a_{10}b_5c_4 + 2a_1a_5a_{10}b_5c_5 + 2a_1a_5a_{14}b_6c_4 + 2a_1a_5a_{14}b_6c_6 + 2a_1a_8a_9b_5c_3 + 2a_1a_8a_{10}b_5c_4 + 2a_1a_8a_{15}b_6c_5 + 2a_1a_8a_{15}b_6c_6 + 2a_1a_{12}a_{13}b_6c_3 + 2a_1a_{12}a_{14}b_6c_4 + 2a_1a_{12}a_{15}b_6c_5 + a_6^2b_4c_3^2 + a_6a_{10}b_5c_3^2 + 2a_6a_{10}b_5c_3c_4 + 2a_6a_{10}b_5c_3c_5 + a_6a_{14}b_6c_3^2 + 2a_6a_{14}b_6c_3c_4 + 2a_6a_{14}b_6c_3c_6 + a_9^2b_5c_3^2 + 2a_9a_{10}b_5c_3c_4 + a_9a_{15}b_6c_3^2 + 2a_9a_{15}b_6c_3c_5 + 2a_9a_{15}b_6c_3c_6 + a_{10}^2b_5c_4^2 + a_{10}a_{15}b_6c_4^2 + 2a_{10}a_{15}b_6c_4c_5 + 2a_{10}a_{15}b_6c_4c_6 + a_{13}^2b_6c_3^2 + 2a_{13}a_{14}b_6c_3c_4 + 2a_{13}a_{15}b_6c_3c_5 + a_{14}^2b_6c_4^2 + 2a_{14}a_{15}b_6c_4c_5 + a_{15}^2b_6c_5^2 = \frac{11}{60}$$

(26i)

$$a_1^3a_3b_3 + a_1^3a_5b_4 + a_1^3a_8b_5 + a_1^3a_{12}b_6 + 3a_1a_3b_3c_3^2 + 3a_1a_5b_4c_4^2 + 3a_1a_8b_5c_5^2 + 3a_1a_{12}b_6c_6^2 + a_6b_4c_3^3 + 3a_6b_4c_3c_4^2 + a_9b_5c_3^3 + 3a_9b_5c_3c_5^2 + a_{10}b_5c_4^3 + 3a_{10}b_5c_4c_5^2 + a_{13}b_6c_3^3 + 3a_{13}b_6c_3c_6^2 + a_{14}b_6c_4^3 + 3a_{14}b_6c_4c_6^2 + a_{15}b_6c_5^3 + 3a_{15}b_6c_5c_6^2 = \frac{7}{20}$$

(26j)

$$a_1^2a_3b_3c_3 + a_1^2a_5b_4c_4 + a_1^2a_8b_5c_5 + a_1^2a_{12}b_6c_6 + a_6b_4c_3^2c_4 + a_9b_5c_3^2c_5 + a_{10}b_5c_4^2c_5 + a_{13}b_6c_3^2c_6 + a_{14}b_6c_4^2c_6 + a_{15}b_6c_5^2c_6 = \frac{1}{15}$$

(26k)

$$a_1^4b_2 + b_3c_3^4 + b_4c_4^4 + b_5c_5^4 + b_6c_6^4 = \frac{1}{5}$$

(26l)

Note, for convenience, we assume values for the following parameters;

$$\left.\begin{aligned} &c_1 = 0, c_2 = a_1 = \frac{1}{3}, c_3 = (a_2 + a_3) = \frac{2}{3}, c_4 = (a_4 + a_5 + a_6) = \frac{1}{3}, c_5 = (a_7 + a_8 + a_9 + a_{10}) = \frac{3}{5} \\ &c_6 = (a_{11} + a_{12} + a_{13} + a_{14} + a_{15}) = 1, \text{ and } b_2 = \frac{1}{3}; \end{aligned}\right\}$$

(27)

From (26a), (26b), (26d), (26g) and (26l), we solve simultaneously with the help of Maple-18 software and obtain the following results:

$$b_1 = \frac{7}{72}, b_3 = \frac{9}{8}, b_4 = \frac{11}{48}, b_5 = -\frac{125}{144}, b_6 = \frac{1}{12}$$

By substituting $b_1, b_2, b_3, b_4, b_5, b_6, c_3, c_4, c_5, c_6$ and a_1 into the remaining seven equations, we observed that there are seven equations with ten unknown which is not practically solvable. Thus we have three "free" parameters in other to solve the equations. Hence, setting $a_3 = \frac{2}{3}, a_{14} = -\frac{7}{5}, a_{15} = \frac{3}{5},$ we have

$$a_5 = \frac{43549}{7217},\ a_6 = -\frac{30361}{14840}, a_8 = \frac{35525}{9169},\ a_9 = -\frac{27646}{19955},\ a_{10} = -\frac{10643}{155037},\ a_{12} = \frac{736810}{53619}, a_{13} = -\frac{28702}{5227}$$

$$a_2 = 0,\ a_4 = -\frac{167765027}{45900120},\ a_7 = -\frac{51638854921283}{28366716018615},\ a_{11} = -\frac{9039268043}{1401332565}$$

Substituting these values of the $a_{i's}$ above in the general Runge – Kutta method we have the new formula becomes:

$$y_{n+1} - y_n = \frac{h}{144}(14k_1 + 48k_2 + 162k_3 + 33k_4 - 125k_5 + 12k_6) \tag{28}$$

Where

$$k_1 = f(y_n)$$

$$k_2 = f(y_n + \frac{1}{3}hk_1)$$

$$k_3 = f\left(y_n + \frac{2}{3}hk_2\right)$$

$$k_4 = f\left(y_n + h\left(-\frac{167765027}{45900120}k_1 + \frac{43549}{7217}k_2 - \frac{30361}{14840}k_3\right)\right)$$

$$k_5 = f\left(y_n + h\left(-\frac{51638854921283}{28366716018615}k_1 + \frac{35525}{9169}k_2 - \frac{27646}{19955}k_3 - \frac{10643}{155037}k_4\right)\right)$$

$$k_6 = f\left(y_n + h\left(-\frac{9039268043}{1401332565}k_1 + \frac{736810}{53619}k_2 - \frac{28702}{5227}k_3 - \frac{7}{5}k_4 + \frac{3}{5}k_5\right)\right)$$

The Butcher's array will be;

$$
\begin{array}{c|cccccc}
0 & 0 & & & & & \\
\frac{1}{3} & \frac{1}{3} & & & & & \\
\frac{2}{3} & 0 & \frac{2}{3} & & & & \\
\frac{1}{3} & -\frac{167765027}{45900120} & \frac{43549}{7217} & -\frac{30361}{14840} & & & \\
\frac{3}{5} & -\frac{51638854921283}{28366716018615} & \frac{35525}{9169} & -\frac{27646}{19955} & -\frac{10643}{155037} & & \\
1 & -\frac{9039268043}{1401332565} & \frac{736810}{53619} & -\frac{28702}{5227} & -\frac{7}{5} & \frac{3}{5} & \\
\hline
 & \frac{7}{72} & \frac{1}{3} & \frac{9}{8} & \frac{11}{48} & -\frac{125}{144} & \frac{1}{12}
\end{array}
$$

III. Theorem

the new Sixth-Stage Fifth-Order Runge-Kutta formula is consistent and converges very fast for any initial value problems of the form $y' = f(x,y)$, $y(x_0) = y_0$, $i.e$ $\phi(x,y,0) = f(x,y)$

Proof:

Using equation (28) with $k_{i's}$ as given in equations (28a) to (28f) we establish that the new formula is convergent and consistent in the solution of the ivp, $y' = f(x,y)$, $y(x_0) = y_0$, $i.e$ $\phi(x,y,0) = f(x,y)$

Given that:

$$
\begin{aligned}
&T_n(h^6) = y_{n+1} - y_n = \frac{h}{144}(14f(y_n) + 48[f(y_n + a_1 h(f(y_n)))] + \\
&162[f(y_n + h(a_2 f(y_n) + a_3(f(y_n + a_1 hf(y_n)))))] + \\
&33[f(y_n + h(a_4 f(y_n) + a_5(f(y_n + a_1 hf(y_n))) + a_6(f(y_n + h(a_2 f(y_n)) \\
&+a_3(f(y_n + a_1 hf(y))))))))] - 125[f(y_n + h(a_7 f(y_n) + a_8(f(y_n + a_1 hf(y_n))) \\
&+a_9(f(y_n + h(a_2 f(y_n) + a_3(f(y_n + a_1 hf(y_n))))))))] \\
&+12[f(y_n + h(a_{11} f(y_n) + a_{12}(f(y_n + a_1 hf(y_n))) \\
&+a_{13}(f(y_n + h(a_2 f(y_n) + h(a_2 f(y_n) + a_3(f(y_n + a_1 hf(y_n)))))) + \\
&a_{14}(f(y_n + h(a_4 f(y_n) + a_5(f(y_n + a_1 hf(y_n))) + \\
&a_6(f(y_n + h(a_2 f(y_n) + a_3(f(y_n + a_1 hf(y_n)))))))) \\
&+a_{15}(f(y_n + h(a_7 f(y_n) + a_8(f(y_n + a_1 hf(y_n))) + a_9(f(y_n + h(a_2 f(y_n)) \\
&+a_3(f(y_n + a_1 hf(y_n)))))) \\
&+a_{10}(f(y_n + h(a_4 f(y_n) + a_5(f(y_n + a_1 hf(y_n))) + \\
&a_6(f(y_n + h(a_2 f(y_n) + a_3(f(y_n) + a_1 hf(y_n)))))))))))]
\end{aligned}
$$

Dividing all through by h and taking limits as $h \to 0$, we obtain;

$$\left(\frac{y_{n+1}-y_n}{h}\right)=\frac{h}{144}(14f(y_n)+48f(y_n)+162f(y_n)+33f(y_n)-125f(y_n)+12f(y_n))$$

$$\left(\frac{y_{n+1}-y_n}{h}\right)=\frac{1}{144}(144f(y_n))$$

$$\left(\frac{y_{n+1}-y_n}{h}\right)=y^{'}=f(y_n)$$

Hence the method is consistent with the solution of the initial value problem. According to Lambert [12] the consistency of a method invariably implies convergence.

IV. IMPLEMENTATION OF THE NEW FORMULA

In order to prove the usefulness of our method, comparisons were made with the Kutta-Nystrom method, by solving some selected initial value problems as shown below. The numerical solution to these initial value problems were generated by using MATLAB package.

Problem 1: $y^{'}=y;\quad y(0)=1,\quad 0\le x\le 1$ (Lambert, [10])

With theoretical solution $y(x)=e^{x}$, $h=0.1$

Problem 2: $y^{'}=y+1;\quad y(0)=1,\quad 0\le x\le 1$ With theoretical solution $y(x)=-1+2e^{x}$, $h=0.1$ **(**Agbeboh [4])

Problem 3: $y^{'}=y^{2}+1;\quad y(0)=1,\quad 0\le x\le 1$ (Lambert [10])

With theoretical solution $y(x)=\tan\left(x+\frac{1}{4}\pi\right),\quad h=0.1$

TABLE 1; NUMERICAL RESULT FOR PROBLEM 1

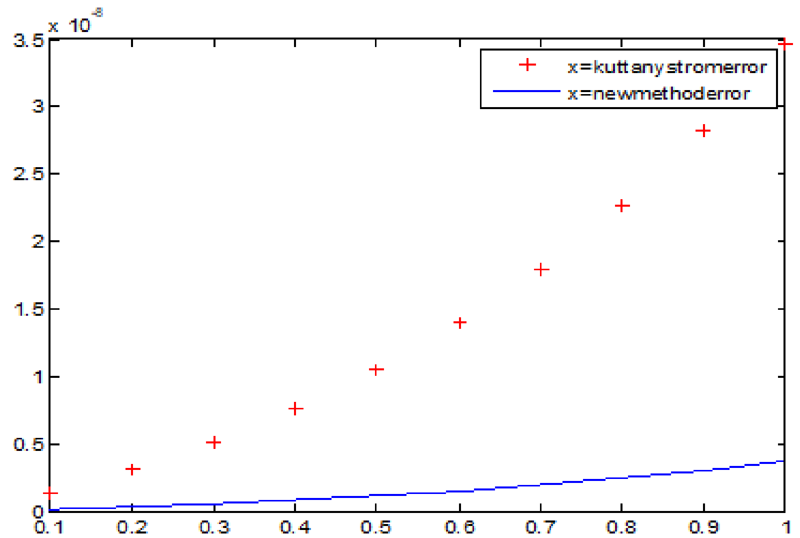
		NEW METHOD		KUTTA-NYSTROM METHOD	
XN	TSOL	YN	ERROR	YN	ERROR
0.1	1.105170918076	1.105170918229	1.5291146127E-10	1.105170916667	1.4089811540E-09
0.2	1.221402758160	1.221402758498	3.3798674970E-10	1.221402755046	3.1143296830E-09
0.3	1.349858807576	1.349858808136	5.6029958451E-10	1.349858802413	5.1627999653E-09
0.4	1.491824697641	1.491824698467	8.2563555992E-10	1.491824690034	7.6077015798E-09
0.5	1.648721270700	1.648721271841	1.1405854039E-09	1.648721260190	1.0509763060E-08
0.6	1.822118800391	1.822118801903	1.5126502273E-09	1.822118786452	1.3938101118E-08
0.7	2.013752707470	2.013752709421	1.9503594295E-09	2.013752689499	1.7971314659E-08
0.8	2.225540928492	2.225540930956	2.4634068119E-09	2.225540905794	2.2698713131E-08
0.9	2.459603111157	2.459603114220	3.0627962389E-09	2.459603082935	2.8221702308E-08
1	2.718281828459	2.718281832220	3.7610146109E-09	2.718281793804	3.4655338599E-08

TABLE 2; NUMERICAL RESULT FOR PROBLEM 2

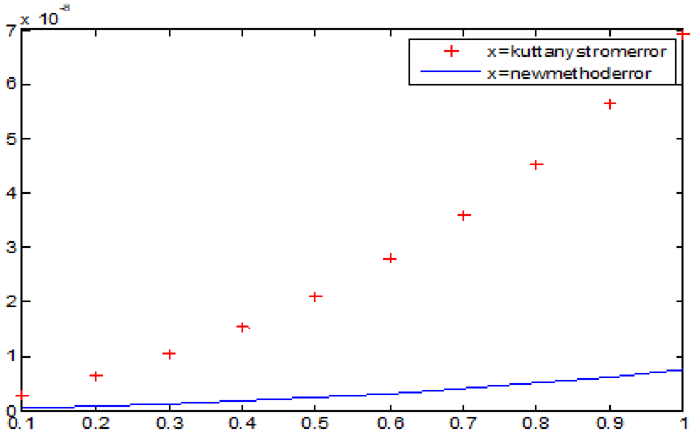
	NEW METHOD (RKM(6, 5))			KUTTA-NYSTROM METHOD (RKM(6, 5))	
XN	TSOL	YN	ERROR	YN	ERROR
0.1	1.210341836151	1.210341836457	3.0582270050E-10	1.210341833333	2.8179620859E-09
0.2	1.442805516320	1.442805516996	6.7597305531E-10	1.442805510092	6.2286591440E-09
0.3	1.699717615152	1.699717616273	1.1205985029E-09	1.699717604826	1.0325599487E-08
0.4	1.983649395283	1.983649396934	1.6512706758E-09	1.983649380067	1.5215402938E-08
0.5	2.297442541400	2.297442543681	2.2811703637E-09	2.297442520381	2.1019526120E-08
0.6	2.644237600781	2.644237603806	3.0253004546E-09	2.644237572905	2.7876202235E-08
0.7	3.027505414941	3.027505418842	3.9007193031E-09	3.027505378998	3.5942629761E-08
0.8	3.451081856985	3.451081861912	4.9268140678E-09	3.451081811588	4.5397427151E-08
0.9	3.919206222314	3.919206228439	6.1255933659E-09	3.919206165870	5.6443405505E-08
1	4.436563656918	4.436563664440	7.5220301099E-09	4.436563587607	6.9310678086E-08

TABLE 3; NUMERICAL RESULT FOR PROBLEM 3

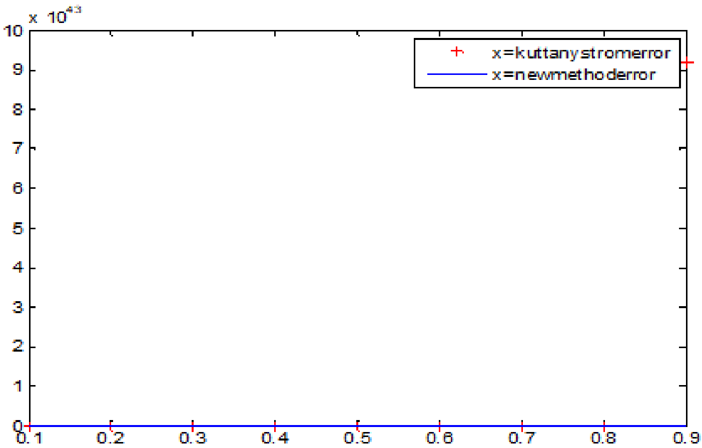
		NEW METHOD		KUTTA-NYSTROM METHOD	
XN	TSOL	YN	ERROR	YN	ERROR
0.1	1.2230488804	1.22304202196	6.8584838842E-06	1.223049093774	2.1332441102E-07
0.2	1.5084976471	1.508471238020	2.6409101154E-05	1.508498450204	8.0308232087E-07
0.3	1.8957651228	1.895678730049	8.6392805462E-05	1.895767712257	2.5894027695E-06
0.4	2.4649627567	2.464667366440	2.9539028262E-04	2.464971747838	8.9911157395E-06
0.5	3.4082234423	3.407047937139	1.1755051964E-03	3.408262855821	3.9413485211E-05
0.6	5.3318552234	5.32580354661	6.0516768469E-03	5.332122402518	2.6717905900E-04
0.7	11.6813738003	11.60415254436	7.7221255951E-02	11.681002646740	3.7115356976E-04
0.8	68.4796683455	1544.9815278800	1.6134611962E+03	536.906662712371	6.0538633106E+02
0.9	-8.6876295464	3.834804034177	3.8348040342E+00	9.186661743016	9.1866617430E+43
1	-4.5880378249	NaN	NaN	NaN	NaN



PLOT 1: ERROR ANALYSIS OF PROBLEM 1



PLOT 2: ERROR ANALYSIS OF PROBLEM 2



PLOT 3: ERROR ANALYSIS OF PROBLEM 3

V. SUMARRY AND CONCLUSION

A sixth- stage fifth-order Runge-Kutta formula for solving initial value problems in ordinary differential equation was derived and implemented. A careful look at the tables of results for the solved problems revealed the effectiveness of the Sixth-Stage Fifth-Order Runge-Kutta's methods by displaying relatively low error level, when compared with the result from Kutta-Nystrom method for the same ivps.

It was also observed that the new method has the capacity to solve linear and non-linear problems, which can be seen from the problems solved above. This new method maintains a high degree of accuracy in handling first order initial value problems. We hope to extend the method to second order initial value problems with a view to finding out the level of stability of the method. From the results obtained via the numerical experiment, it shows that the method is consistent and appropriate for the solution of non-stiff initial value problems in Ordinary Differential Equation.

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