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The Derivation and Implementation of Order 7 Rational Interpolation Scheme for Solving Initial Value Problems in Ordinary Differential Equation

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ABSTRACT: *A new rational interpolation method of order 7 is derived with the use MAPLE-18 software. The resulting scheme is use to solve some selected initial value problems, and comparisons were conducted to ascertain the level of performance of the method.*

Keywords: *Rational integrator, stiff-problem, Determinant and Ordinary Differential Equation.*

I. INTRODUCTION

Numerical methods for evaluating systems of Ordinary Differential Equations (ODEs) have been attracting much attention because they proffer the solutions of problems arising from the mathematical formulation of physical situations such as those in chemical kinetics, population, economic, political and social models. Numerical solution of ordinary differential equations can be obtained using rational integrators, such as linear multistep, Runge-kutta and exponential methods and many others. There are quite a number of stiff problems that can be solved by rational integrators, which are from physical reactions, chemical kinetic and life sciences. The works of Aashikpelokhai (1991, 2000), Lambert (1973), Fatunla (1982), and Agbeboh (2006) established that rational integrator methods, can solve these problems. Agbeboh and Aashikpelokhai (2002) worked on implementation of rational integrator, of order 26, which produced result that compared favorably well with other existing methods without really establishing the stability region of the method Our computational experience as exemplified by the works of Fatunla and Aashikpelokhai (1994), Aashikpelokhai and Elakhe (2010) along with the research work given by Otunta and Ikhile (1996, 1999) all give credence to the need for rational integrators. . All of these works were basically for the improvement of numerical solutions to initial value problems, which sets the foundation for the study.

II. DERIVATION OF METHOD

The general rational interpolation method is defined as:

$$U(x) = \frac{\sum_{r=0}^{3} p_r x^r}{1+\sum_{r=1}^{4} q_r x^r} \tag{2.1}$$

where p_0, p_1, p_2, p_3 and q_1, q_2, q_3 and q_4 are integrator parameters to be determined.
But

$$U(x) = \sum_{r=0}^{\infty} c_r x^r \tag{2.2}$$

Hence (2.1) becomes

$$\sum_{r=0}^{\infty} c_r x^r \left(1+\sum_{r=1}^{4} q_r x^r\right) = \sum_{r=0}^{3} p_r x^r \tag{2.3}$$

On summing up, we obtain:

$$\left(x^{10}c_{10}+x^{9}c_{9}+x^{8}c_{8}+x^{7}c_{7}+x^{6}c_{6}+x^{5}c_{5}+x^{4}c_{4}+x^{3}c_{3}+x^{2}c_{2}+xc_{1}+c_{0}\right)\left(x^{4}q_{4}+x^{3}q_{3}+x^{2}q_{2}+xq_{1}+1\right)=x^{3}p_{3}+x^{2}p_{2}+xp_{1}+p_{0} \quad (2.4)$$

On expanding, we have:

$$x^{14}c_{10}q_{4}+\left(c_{9}q_{4}+c_{10}q_{3}\right)x^{13}+\left(c_{8}q_{4}+c_{9}q_{3}+c_{10}q_{2}\right)x^{12}+\left(c_{7}q_{4}+c_{8}q_{3}+c_{9}q_{2}+c_{10}q_{1}\right)x^{11}+\left(c_{6}q_{4}+c_{7}q_{3}+c_{8}q_{2}+c_{9}q_{1}+c_{10}\right)x^{10}+\left(c_{5}q_{4}+c_{6}q_{3}+c_{7}q_{2}+c_{8}q_{1}+c_{9}\right)x^{9}+\left(c_{4}q_{4}+c_{5}q_{3}+c_{6}q_{2}+c_{7}q_{1}+c_{8}\right)x^{8}+\left(c_{3}q_{4}+c_{4}q_{3}+c_{5}q_{2}+c_{6}q_{1}+c_{7}\right)x^{7}+\left(c_{2}q_{4}+c_{3}q_{3}+c_{4}q_{2}+c_{5}q_{1}+c_{6}\right)x^{6}+\left(c_{1}q_{4}+c_{2}q_{3}+c_{3}q_{2}+c_{4}q_{1}+c_{5}\right)x^{5}+\left(c_{0}q_{4}+c_{1}q_{3}+c_{2}q_{2}+c_{3}q_{1}+c_{4}\right)x^{4}+\left(c_{0}q_{3}+c_{1}q_{2}+c_{2}q_{1}+c_{3}\right)x^{3}+\left(c_{0}q_{2}+c_{1}q_{1}+c_{2}\right)x^{2}+\left(c_{0}q_{1}+c_{1}\right)x+c_{0}=x^{3}p_{3}+x^{2}p_{2}+xp_{1}+p_{0} \quad (2.5)$$

Collecting the coefficients of x, we have:

$$\begin{aligned}
p_0 &= c_0\\
p_1 &= \left(c_0q_1+c_1\right)\\
p_2 &= \left(c_0q_2+c_1q_1+c_2\right)\\
p_3 &= \left(c_0q_3+c_1q_2+c_2q_1+c_3\right):\\
p_4 &= \left(c_0q_4+c_1q_3+c_2q_2+c_3q_1+c_4\right)\\
p_5 &= \left(c_1q_4+c_2q_3+c_3q_2+c_4q_1+c_5\right)\\
p_6 &= \left(c_2q_4+c_3q_3+c_4q_2+c_5q_1+c_6\right)\\
p_7 &= \left(c_3q_4+c_4q_3+c_5q_2+c_6q_1+c_7\right)
\end{aligned} \quad (2.6)$$

Since the numerator of (2.1) is order 3, then the integrator parameter $p_4, p_5, p_6, p_7 = 0$

Hence

$$\begin{aligned}
\left(c_0q_4+c_1q_3+c_2q_2+c_3q_1\right) &= c_4\\
\left(c_1q_4+c_2q_3+c_3q_2+c_4q_1\right) &= c_5\\
\left(c_2q_4+c_3q_3+c_4q_2+c_5q_1\right) &= c_6\\
\left(c_3q_4+c_4q_3+c_5q_2+c_6q_1\right) &= c_7
\end{aligned} \quad (2.7)$$

Rearranging and putting (2.7) in matrix form, gives:

$$\begin{bmatrix} c_6 & c_5 & c_4 & c_3\\ c_5 & c_4 & c_3 & c_2\\ c_4 & c_3 & c_2 & c_1\\ c_3 & c_2 & c_1 & c_0 \end{bmatrix}\begin{bmatrix} q_7\\ q_6\\ q_5\\ q_4 \end{bmatrix}=\begin{bmatrix} c_7\\ c_6\\ c_5\\ c_4 \end{bmatrix} \quad (2.8)$$

The above equation which can be put in the form:

$$Ax=b \quad (2.9)$$

is the governing equation that will be used to derive the new method.

Now, to solve for q_1, q_2, q_3 and q_4, we use the crammer's rule which is given as:

$$q_i=\frac{x_i}{|A|}, \text{ where } i=1.2.3.4. \quad (2.10)$$

First, we construct the determinants from (2.8) as follows:

$$A = \begin{bmatrix} c_6 & c_5 & c_4 & c_3 \\ c_5 & c_4 & c_3 & c_2 \\ c_4 & c_3 & c_2 & c_1 \\ c_3 & c_2 & c_1 & c_0 \end{bmatrix} \tag{2.11}$$

$$x_1 = \begin{bmatrix} -c_7 & c_5 & c_4 & c_3 \\ -c_6 & c_4 & c_3 & c_2 \\ -c_5 & c_3 & c_2 & c_1 \\ -c_4 & c_2 & c_1 & c_0 \end{bmatrix} \tag{2.12}$$

$$x_2 = \begin{bmatrix} c_6 & -c_7 & c_4 & c_3 \\ c_5 & -c_6 & c_3 & c_2 \\ c_4 & -c_5 & c_2 & c_1 \\ c_3 & -c_4 & c_1 & c_0 \end{bmatrix} \tag{2.13}$$

$$x_3 = \begin{bmatrix} c_6 & c_5 & -c_7 & c_3 \\ c_5 & c_4 & -c_6 & c_2 \\ c_4 & c_3 & -c_5 & c_1 \\ c_3 & c_2 & -c_4 & c_0 \end{bmatrix} \tag{2.14}$$

$$x_4 = \begin{bmatrix} c_6 & c_5 & c_4 & -c_7 \\ c_5 & c_4 & c_3 & -c_6 \\ c_4 & c_3 & c_2 & -c_5 \\ c_3 & c_2 & c_1 & -c_4 \end{bmatrix} \tag{2.15}$$

On solving the determinants and simplifying, we get:

$$\begin{aligned} |A| = {} & c_0 c_2 c_4 c_6 - c_0 c_2 c_5^2 - c_0 c_3^2 c_6 + 2 c_0 c_3 c_4 c_5 - c_0 c_4^3 - c_1^2 c_4 c_6 + c_1^2 c_5^2 \\ & + 2 c_1 c_2 c_3 c_6 - 2 c_1 c_2 c_4 c_5 - 2 c_1 c_3^2 c_5 + 2 c_1 c_3 c_4^2 - c_2^3 c_6 + 2 c_2^2 c_3 c_5 + c_2^2 c_4^2 \\ & - 3 c_2 c_3^2 c_4 + c_3^4 \end{aligned} \tag{2.16}$$

$$\begin{aligned} |x_1| = {} & -c_0 c_2 c_4 c_7 + c_0 c_2 c_5 c_6 + c_0 c_3^2 c_7 - c_0 c_3 c_4 c_6 - c_0 c_3 c_5^2 + c_0 c_4^2 c_5 + \\ & c_1^2 c_4 c_7 - c_1^2 c_5 c_6 - 2 c_1 c_2 c_3 c_7 + c_1 c_2 c_4 c_6 + c_1 c_2 c_5^2 + c_1 c_3^2 c_6 - c_1 c_4^3 + \\ & c_2^3 c_7 - c_2^2 c_3 c_6 - 2 c_2^2 c_4 c_5 + c_2 c_3^2 c_5 + 2 c_2 c_3 c_4^2 - c_3^3 c_4 \end{aligned} \tag{2.17}$$

At this point, taking $x = x_{n+1}$, (2.2) becomes:

$$y_{n+1} = \sum_{r=0}^{\infty} c_r x_{n+1}^r \qquad (2.18)$$

So by Taylor series expansion of y_{n+1}, we have:

$$y_{n+1} = \sum_{r=0}^{\infty} \frac{h^r y_n^{(r)}}{r!} \qquad (2.19)$$

Therefore,

$$\sum_{r=0}^{\infty} c_r x_{n+1}^r = \sum_{r=0}^{\infty} \frac{h^r y_n^{(r)}}{r!} \qquad (2.20)$$

Expanding (2.1.20), we have:

$$c_0 = y_n;\ c_1 = \frac{h^1 \cdot y_n^{(1)}}{1!\cdot x_{n+1}^1};\ c_2 = \frac{h^2 \cdot y_n^{(2)}}{2!\cdot x_{n+1}^2};\ c_3 = \frac{h^3 \cdot y_n^{(3)}}{3!\cdot x_{n+1}^3};\ c_4 = \frac{h^4 \cdot y_n^{(4)}}{4!\cdot x_{n+1}^4};\ c_5 = \frac{h^5 \cdot y_n^{(5)}}{5!\cdot x_{n+1}^5};\ c_6 = \frac{h^6 \cdot y_n^{(6)}}{6!\cdot x_{n+1}^6};\ c_7 = \frac{h^7 \cdot y_n^{(7)}}{7!\cdot x_{n+1}^7}; \qquad (2.21)$$

With (2.21), the determinant of A in (2.11) can be written as:

$$|A| = -\frac{h^{12}}{1036800\cdot x_{n+1}^{12}}\cdot\Big(72\, y_n^{(1)^2} y_n^{(5)^2} - 36\, y_n y_n^{(2)} y_n^{(5)^2} - 360\, y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(5)} - 480\, y_n^{(1)} y_n^{(3)^2} y_n^{(5)} + 720\, y_n^{(2)^2} y_n^{(3)} y_n^{(5)} + 120\, y_n y_n^{(3)} y_n^{(4)} y_n^{(5)} - 60\, y_n^{(1)^2} y_n^{(4)} y_n^{(6)} + 240\, y_n^{(1)} y_n^{(2)} y_n^{(3)} y_n^{(6)} + 600\, y_n^{(1)} y_n^{(3)} y_n^{(4)^2} - 180\, y_n^{(2)^3} y_n^{(6)} + 450\, y_n^{(2)^2} y_n^{(4)^2} + 30\, y_n y_n^{(2)} y_n^{(4)} y_n^{(6)} - 1800\, y_n^{(2)} y_n^{(3)^2} y_n^{(4)} - 40\, y_n y_n^{(3)^2} y_n^{(6)} - 75\, y_n y_n^{(4)^3} + 800\, y_n^{(3)^4}\Big) \qquad (2.22)$$

$$|x_1| := -\frac{h^{13}}{7257600\cdot x_{n+1}^{13}}\cdot\Big(84\, y_n y_n^{(3)} y_n^{(5)^2} - 252\, y_n^{(1)} y_n^{(2)} y_n^{(5)^2} - 840\, y_n^{(2)} y_n^{(3)^2} y_n^{(5)} + 84\, y_n^{(1)^2} y_n^{(5)} y_n^{(6)} - 42\, y_n y_n^{(2)} y_n^{(5)} y_n^{(6)} + 1260\, y_n^{(2)^2} y_n^{(4)} y_n^{(5)} - 105\, y_n y_n^{(4)^2} y_n^{(5)} + 1400\, y_n^{(3)^3} y_n^{(4)} - 280\, y_n^{(1)} y_n^{(3)^2} y_n^{(6)} - 40\, y_n y_n^{(3)^2} y_n^{(7)} + 420\, y_n^{(2)^2} y_n^{(3)} y_n^{(6)} + 70\, y_n y_n^{(3)} y_n^{(4)} y_n^{(6)} + 240\, y_n^{(1)} y_n^{(2)} y_n^{(3)} y_n^{(7)} - 2100\, y_n^{(2)} y_n^{(3)} y_n^{(4)^2} - 210\, y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(6)} - 60\, y_n^{(1)^2} y_n^{(4)} y_n^{(7)} + 525\, y_n^{(1)} y_n^{(4)^3} - 180\, y_n^{(2)^3} y_n^{(7)} + 30\, y_n y_n^{(2)} y_n^{(4)} y_n^{(7)}\Big): \qquad (2.23)$$

$$|x_2| = -\frac{h^{14}}{14515200\cdot x_{n+1}^{14}}\cdot\Big(120\, y_n^{(2)^2} y_n^{(3)} y_n^{(7)} - 210\, y_n^{(2)^2} y_n^{(4)} y_n^{(6)} - 12\, y_n y_n^{(2)} y_n^{(5)} y_n^{(7)} + 84\, y_n^{(1)} y_n^{(2)} y_n^{(5)} y_n^{(6)} - 60\, y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(7)} + 14\, y_n y_n^{(2)} y_n^{(6)^2} - 280\, y_n^{(2)} y_n^{(3)^2} y_n^{(6)} + 525\, y_n^{(2)} y_n^{(4)^3} - 168\, y_n^{(1)} y_n^{(3)} y_n^{(5)^2} + 42\, y_n y_n^{(4)} y_n^{(5)^2} + 24\, y_n^{(1)^2} y_n^{(5)} y_n^{(7)} - 28\, y_n y_n^{(3)} y_n^{(5)} y_n^{(6)} + 560\, y_n^{(3)^3} y_n^{(5)} - 210\, y_n^{(1)} y_n^{(4)^2} y_n^{(5)} - 80\, y_n^{(1)} y_n^{(3)^2} y_n^{(7)} + 20\, y_n y_n^{(3)} y_n^{(4)} y_n^{(7)} - 28\, y_n^{(1)^2} y_n^{(6)^2} + 420\, y_n^{(1)} y_n^{(3)} y_n^{(4)} y_n^{(6)} - 35\, y_n y_n^{(4)^2} y_n^{(6)} - 700\, y_n^{(3)^2} y_n^{(4)^2}\Big): \qquad (2.24)$$

$$
|x_3| \frac{h^{15}}{217728000 \cdot x_{n+1}^{15}} \cdot \Big(1400 {y_n^{(3)}}^3 y_n^{(6)} - 4200 {y_n^{(3)}}^2 y_n^{(4)} y_n^{(5)} - 600 y_n^{(2)} {y_n^{(3)}}^2 y_n^{(7)} + 2520 y_n^{(2)} y_n^{(3)} {y_n^{(5)}}^2 + 60 y_n y_n^{(3)} y_n^{(5)} y_n^{(7)}
- 420 y_n^{(1)} y_n^{(3)} y_n^{(5)} y_n^{(6)} + 300 y_n^{(1)} y_n^{(3)} y_n^{(4)} y_n^{(7)} - 70 y_n y_n^{(3)} {y_n^{(6)}}^2 + 2625 y_n^{(3)} {y_n^{(4)}}^3 - 126 y_n {y_n^{(5)}}^3 + 630 y_n^{(1)} y_n^{(4)} {y_n^{(5)}}^2
- 180 y_n^{(1)} y_n^{(2)} y_n^{(5)} y_n^{(7)} + 210 y_n y_n^{(4)} y_n^{(5)} y_n^{(6)} - 630 {y_n^{(2)}}^2 y_n^{(5)} y_n^{(6)} - 1575 y_n^{(2)} {y_n^{(4)}}^2 y_n^{(5)} - 75 y_n {y_n^{(4)}}^2 y_n^{(7)} + 450 {y_n^{(2)}}^2 y_n^{(4)}
$$

(2.25)

$$
|x_4| = -\frac{h^{12}}{1036800 \cdot x_{n+1}^{12}} \cdot \Big(72 {y_n^{(1)}}^2 {y_n^{(5)}}^2 - 36 y_n y_n^{(2)} {y_n^{(5)}}^2 - 360 y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(5)} - 480 y_n^{(1)} {y_n^{(3)}}^2 y_n^{(5)} + 720 {y_n^{(2)}}^2 y_n^{(3)} y_n^{(5)}
+ 120 y_n y_n^{(3)} y_n^{(4)} y_n^{(5)} - 60 {y_n^{(1)}}^2 y_n^{(4)} y_n^{(6)} + 240 y_n^{(1)} y_n^{(2)} y_n^{(3)} y_n^{(6)} + 600 y_n^{(1)} y_n^{(3)} {y_n^{(4)}}^2 - 180 {y_n^{(2)}}^3 y_n^{(6)} + 450 {y_n^{(2)}}^2 {y_n^{(4)}}^2
+ 30 y_n y_n^{(2)} y_n^{(4)} y_n^{(6)} - 1800 y_n^{(2)} {y_n^{(3)}}^2 y_n^{(4)} - 40 y_n {y_n^{(3)}}^2 y_n^{(6)} - 75 y_n {y_n^{(4)}}^3 + 800 {y_n^{(3)}}^4 \Big):
$$

(2.26)

From (2.22) to (2.26) above, we have the following:

$$
q_1 x_{n+1} = \frac{h}{7} \cdot \Big(\Big(84 y_n y_n^{(3)} {y_n^{(5)}}^2 - 252 y_n^{(1)} y_n^{(2)} {y_n^{(5)}}^2 - 840 y_n^{(2)} {y_n^{(3)}}^2 y_n^{(5)} + 84 {y_n^{(1)}}^2 y_n^{(5)} y_n^{(6)} - 42 y_n y_n^{(2)} y_n^{(5)} y_n^{(6)} + 1260
{y_n^{(2)}}^2 y_n^{(4)} y_n^{(5)} - 105 y_n {y_n^{(4)}}^2 y_n^{(5)} + 1400 {y_n^{(3)}}^3 y_n^{(4)} - 280 y_n^{(1)} {y_n^{(3)}}^2 y_n^{(6)} - 40 y_n {y_n^{(3)}}^2 y_n^{(7)} + 420 {y_n^{(2)}}^2 y_n^{(3)} y_n^{(6)} + 70 y_n y_n^{(3)}
y_n^{(4)} y_n^{(6)} + 240 y_n^{(1)} y_n^{(2)} y_n^{(3)} y_n^{(7)} - 2100 y_n^{(2)} y_n^{(3)} {y_n^{(4)}}^2 - 210 y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(6)} - 60 {y_n^{(1)}}^2 y_n^{(4)} y_n^{(7)} + 525 y_n^{(1)} {y_n^{(4)}}^3 - 180
{y_n^{(2)}}^3 y_n^{(7)} + 30 y_n y_n^{(2)} y_n^{(4)} y_n^{(7)} \Big) \Big) \Big/ \Big(\Big(72 {y_n^{(1)}}^2 {y_n^{(5)}}^2 - 36 y_n y_n^{(2)} {y_n^{(5)}}^2 - 360 y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(5)} - 480 y_n^{(1)} {y_n^{(3)}}^2 y_n^{(5)}
+ 720 {y_n^{(2)}}^2 y_n^{(3)} y_n^{(5)} + 120 y_n y_n^{(3)} y_n^{(4)} y_n^{(5)} - 60 {y_n^{(1)}}^2 y_n^{(4)} y_n^{(6)} + 240 y_n^{(1)} y_n^{(2)} y_n^{(3)} y_n^{(6)} + 600 y_n^{(1)} y_n^{(3)} {y_n^{(4)}}^2 - 180 {y_n^{(2)}}^3
y_n^{(6)} + 450 {y_n^{(2)}}^2 {y_n^{(4)}}^2 + 30 y_n y_n^{(2)} y_n^{(4)} y_n^{(6)} - 1800 y_n^{(2)} {y_n^{(3)}}^2 y_n^{(4)} - 40 y_n {y_n^{(3)}}^2 y_n^{(6)} - 75 y_n {y_n^{(4)}}^3 + 800 {y_n^{(3)}}^4 \Big) \Big):
$$

(2.27)

$$
q_2 \cdot x_{n+1}^2 = \frac{h^2}{14} \cdot \Big(\Big(120 {y_n^{(2)}}^2 y_n^{(3)} y_n^{(7)} - 210 {y_n^{(2)}}^2 y_n^{(4)} y_n^{(6)} - 12 y_n y_n^{(2)} y_n^{(5)} y_n^{(7)} + 84 y_n^{(1)} y_n^{(2)} y_n^{(5)} y_n^{(6)} - 60 y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(7)} + 14 y_n y_n^{(2)} {y_n^{(6)}}^2 - 280
y_n^{(2)} {y_n^{(3)}}^2 y_n^{(6)} + 525 y_n^{(2)} {y_n^{(4)}}^3 - 168 y_n^{(1)} y_n^{(3)} {y_n^{(5)}}^2 + 42 y_n y_n^{(4)} {y_n^{(5)}}^2 + 24 {y_n^{(1)}}^2 y_n^{(5)} y_n^{(7)} - 28 y_n y_n^{(3)} y_n^{(5)} y_n^{(6)} + 560 {y_n^{(3)}}^3 y_n^{(5)} - 210 y_n^{(1)} {y_n^{(4)}}^2 y_n^{(5)}
- 80 y_n^{(1)} {y_n^{(3)}}^2 y_n^{(7)} + 20 y_n y_n^{(3)} y_n^{(4)} y_n^{(7)} - 28 {y_n^{(1)}}^2 {y_n^{(6)}}^2 + 420 y_n^{(1)} y_n^{(3)} y_n^{(4)} y_n^{(6)} - 35 y_n {y_n^{(4)}}^2 y_n^{(6)} - 700 {y_n^{(3)}}^2 {y_n^{(4)}}^2 \Big) \Big) \Big/ \Big(\Big(72 {y_n^{(1)}}^2 {y_n^{(5)}}^2 - 36 y_n
y_n^{(2)} {y_n^{(5)}}^2 - 360 y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(5)} - 480 y_n^{(1)} {y_n^{(3)}}^2 y_n^{(5)} + 720 {y_n^{(2)}}^2 y_n^{(3)} y_n^{(5)} + 120 y_n y_n^{(3)} y_n^{(4)} y_n^{(5)} - 60 {y_n^{(1)}}^2 y_n^{(4)} y_n^{(6)} + 240 y_n^{(1)} y_n^{(2)} y_n^{(3)} y_n^{(6)} + 600
y_n^{(1)} y_n^{(3)} {y_n^{(4)}}^2 - 180 {y_n^{(2)}}^3 y_n^{(6)} + 450 {y_n^{(2)}}^2 {y_n^{(4)}}^2 + 30 y_n y_n^{(2)} y_n^{(4)} y_n^{(6)} - 1800 y_n^{(2)} {y_n^{(3)}}^2 y_n^{(4)} - 40 y_n {y_n^{(3)}}^2 y_n^{(6)} - 75 y_n {y_n^{(4)}}^3 + 800 {y_n^{(3)}}^4 \Big) \Big):
$$

(2.28)

$$
q_3 \cdot x_{n+1}^3 = -\frac{h^3}{210} \cdot \Big(\Big(1400 {y_n^{(3)}}^3 y_n^{(6)} - 4200 {y_n^{(3)}}^2 y_n^{(4)} y_n^{(5)} - 600 y_n^{(2)} {y_n^{(3)}}^2 y_n^{(7)} + 2520 y_n^{(2)} y_n^{(3)} {y_n^{(5)}}^2 + 60 y_n y_n^{(3)} y_n^{(5)} y_n^{(7)}
- 420 y_n^{(1)} y_n^{(3)} y_n^{(5)} y_n^{(6)} + 300 y_n^{(1)} y_n^{(3)} y_n^{(4)} y_n^{(7)} - 70 y_n y_n^{(3)} {y_n^{(6)}}^2 + 2625 y_n^{(3)} {y_n^{(4)}}^3 - 126 y_n {y_n^{(5)}}^3 + 630 y_n^{(1)} y_n^{(4)} {y_n^{(5)}}^2
- 180 y_n^{(1)} y_n^{(2)} y_n^{(5)} y_n^{(7)} + 210 y_n y_n^{(4)} y_n^{(5)} y_n^{(6)} - 630 {y_n^{(2)}}^2 y_n^{(5)} y_n^{(6)} - 1575 y_n^{(2)} {y_n^{(4)}}^2 y_n^{(5)} - 75 y_n {y_n^{(4)}}^2 y_n^{(7)} + 450 {y_n^{(2)}}^2
y_n^{(4)} y_n^{(7)} + 210 y_n^{(1)} y_n^{(2)} {y_n^{(6)}}^2 - 525 y_n^{(1)} {y_n^{(4)}}^2 y_n^{(6)} \Big) \Big) \Big/ \Big(\Big(72 {y_n^{(1)}}^2 {y_n^{(5)}}^2 - 36 y_n y_n^{(2)} {y_n^{(5)}}^2 - 360 y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(5)} - 480
y_n^{(1)} {y_n^{(3)}}^2 y_n^{(5)} + 720 {y_n^{(2)}}^2 y_n^{(3)} y_n^{(5)} + 120 y_n y_n^{(3)} y_n^{(4)} y_n^{(5)} - 60 {y_n^{(1)}}^2 y_n^{(4)} y_n^{(6)} + 240 y_n^{(1)} y_n^{(2)} y_n^{(3)} y_n^{(6)} + 600 y_n^{(1)} y_n^{(3)}
{y_n^{(4)}}^2 - 180 {y_n^{(2)}}^3 y_n^{(6)} + 450 {y_n^{(2)}}^2 {y_n^{(4)}}^2 + 30 y_n y_n^{(2)} y_n^{(4)} y_n^{(6)} - 1800 y_n^{(2)} {y_n^{(3)}}^2 y_n^{(4)} - 40 y_n {y_n^{(3)}}^2 y_n^{(6)} - 75 y_n {y_n^{(4)}}^3 + 800
$$

(2.29)

$$q_4 \cdot x_{n+1}^4 = -\frac{h^4}{840} \cdot \Big(\Big(240\, y_n^{(1)} y_n^{(3)} y_n^{(5)} y_n^{(7)} - 280\, y_n^{(1)} y_n^{(3)} {y_n^{(6)}}^2 - 504\, y_n^{(1)} {y_n^{(5)}}^3 + 840\, y_n^{(1)} y_n^{(4)} y_n^{(5)} y_n^{(6)} - 300\, y_n^{(1)} {y_n^{(4)}}^2 y_n^{(7)} - 800\, {y_n^{(3)}}^3 y_n^{(7)} + 1680\, {y_n^{(3)}}^2 {y_n^{(5)}}^2 + 2800\, {y_n^{(3)}}^2 y_n^{(4)} y_n^{(6)} - 1680\, y_n^{(2)} y_n^{(3)} y_n^{(5)} y_n^{(6)} - 6300\, y_n^{(3)} {y_n^{(4)}}^2 y_n^{(5)} + 1200\, y_n^{(2)} y_n^{(3)} y_n^{(4)} y_n^{(7)} + 2520\, y_n^{(2)} y_n^{(4)} {y_n^{(5)}}^2 - 360\, {y_n^{(2)}}^2 y_n^{(5)} y_n^{(7)} + 420\, {y_n^{(2)}}^2 {y_n^{(6)}}^2 - 2100\, y_n^{(2)} {y_n^{(4)}}^2 y_n^{(6)} + 2625\, {y_n^{(4)}}^4 \Big) \Big) \Big/ \Big(\Big(72\, {y_n^{(1)}}^2 {y_n^{(5)}}^2 - 36\, y_n y_n^{(2)} {y_n^{(5)}}^2 - 360\, y_n^{(1)} y_n^{(2)} y_n^{(4)} y_n^{(5)} - 480\, y_n^{(1)} {y_n^{(3)}}^2 y_n^{(5)} + 720\, {y_n^{(2)}}^2 y_n^{(3)} y_n^{(5)} + 120\, y_n y_n^{(3)} y_n^{(4)} y_n^{(5)} - 60\, {y_n^{(1)}}^2 y_n^{(4)} y_n^{(6)} + 240\, y_n^{(1)} y_n^{(2)} y_n^{(3)} y_n^{(6)} + 600\, y_n^{(1)} y_n^{(3)} {y_n^{(4)}}^2 - 180\, {y_n^{(2)}}^3 y_n^{(6)} + 450\, {y_n^{(2)}}^2 {y_n^{(4)}}^2 + 30\, y_n y_n^{(2)} y_n^{(4)} y_n^{(6)} - 1800\, y_n^{(2)} {y_n^{(3)}}^2 y_n^{(4)} - 40\, y_n {y_n^{(3)}}^2 y_n^{(6)} - 75\, y_n {y_n^{(4)}}^3 + 800\, {y_n^{(3)}}^4 \Big) \Big) : \quad (2.30)$$

Now from (2.1), the rational integrator is expanded to give:

$$U(x) = \frac{p_0 + x p_1 + x^2 p_2 + x^3 p_3}{1 + x q_1 + x^2 q_2 + x^3 q_3 + x^4 q_4}; \quad (2.31)$$

Which can be written as:

$$y_{n+1} = \frac{p_3 x_{n+1}^3 + p_2 x_{n+1}^2 + p_1 x_{n+1} + p_0}{q_4 x_{n+1}^4 + q_3 x_{n+1}^3 + q_2 x_{n+1}^2 + q_1 x_{n+1} + 1} \quad (2.32)$$

With (2.21), we have:

$$p_0 = y_n;$$

$$p_1 := y_n q_1 + \frac{h\, y_n^{(1)}}{x_{n+1}}$$

$$p_2 := y_n q_2 + \frac{h\, y_n^{(1)} q_1}{x_{n+1}} + \frac{1}{2} \frac{h^2 y_n^{(2)}}{x_{n+1}^2} \quad (2.33)$$

$$p_3 := y_n q_3 + \frac{h\, y_n^{(1)} q_2}{x_{n+1}} + \frac{1}{2} \frac{h^2 y_n^{(2)} q_1}{x_{n+1}^2} + \frac{1}{6} \frac{h^3 y_n^{(3)}}{x_{n+1}^3}$$

$$p_0 + p_1 \cdot x_{n+1} + p_2 \cdot x_{n+1}^2 + p_3 \cdot x_{n+1}^3 = y_n + y_n q_1 \cdot x_{n+1} + h\, y_n^{(1)} + y_n q_2 \cdot x_{n+1}^2 + h\, y_n^{(1)} q_1 \cdot x_{n+1} + \frac{1}{2} h^2 y_n^{(2)} + y_n q_3 \cdot x_{n+1}^3 + h\, y_n^{(1)} q_2 \cdot x_{n+1}^2 + \frac{1}{2} h^2\, y_n^{(2)} q_1 \cdot x_{n+1} + \frac{1}{6} h^3 y_n^{(3)} \quad (2.34)$$

On substituting into (2.1), we get:

$$y_{n+1} = \frac{\left(y_n \cdot (1 + A + B + C) + h\, y_n^{(1)} \cdot (1 + A + B) + \frac{1}{2} h^2 y_n^{(2)} \cdot (1 + A) + \frac{1}{6} h^3 y_n^{(3)} \right)}{(1 + A + B + C + D)} \quad (2.35)$$

where $q_1 x_{n+1} = A;\ \ q_2 x_{n+1}^2 = B;\ \ q_3 x_{n+1}^3 = C;\ \ q_4 x_{n+1}^4 = D$

III. IMPLEMENTAION OF METHOD:

In order to ascertain the suitability of our method, we selected two stiff initial value problems and one real life problems, whose solution were provided by our method and compared with other existing method in the literature as given below:.

Problem 1. $y' = -1000y(x) + e^{-2x}; y(0) = 0$

With theoretical solution: $\frac{1}{998} e^{-2x} - \frac{1}{998} e^{-1000x}$

Problem 2. $y' = -8y + 8x + 1;\ \ y(0) = 2;$

With theoretical solution $x + 2\exp(-8x)$

TABLE 1: SOLUTION TO PROBLEM1

		RIM(7)		DIRKM(5,5)	
XN	TSOL	YN	ERROR	YN	ERROR
0.001	0.00063138533	0.000631385	3.80231E-12	0.000631486	1.00474E-07
0.002	0.00086239750	0.000862397	4.97385E-11	0.000862471	7.39114E-08
0.003	0.00094612314	0.000946123	4.81516E-11	0.000946164	4.07764E-08
0.004	0.00097566761	0.000975668	3.09179E-11	0.000975688	1.99943E-08
0.005	0.00098528245	0.000985282	1.65641E-11	0.000985292	9.18890E-09
0.006	0.00098756810	0.000987568	8.03963E-12	0.000987572	4.05158E-09
0.007	0.00098715998	0.000987160	3.66801E-12	0.000987162	1.73424E-09
0.008	0.00098576338	0.000985763	1.59959E-12	0.000985764	7.24535E-10
0.009	0.00098400563	0.000984006	6.69203E-13	0.000984006	2.95281E-10
0.01	0.00098211751	0.000982118	2.65938E-13	0.000982118	1.16113E-10

TABLE 2: SOLUTION TO PROBLEM 2

		RIM(7)		DIRKM(5,5)	
XN	TSOL	YN	ERROR	YN	ERROR
0.01	1.85623269277	1.856232692773	1.79856E-14	1.8562297181	2.97468E-06
0.02	1.72428757793	1.724287577933	3.04201E-14	1.7242820860	5.49194E-06
0.03	1.60325572213	1.603255722133	3.93019E-14	1.6032481176	7.60455E-06
0.04	1.49229807415	1.492298074147	4.50751E-14	1.4922887143	9.35983E-06
0.05	1.39064009207	1.390640092071	4.90719E-14	1.3906292918	1.08003E-05
0.06	1.29756678361	1.297566783612	5.19584E-14	1.2975548198	1.19639E-05
0.07	1.21241812770	1.212418127698	5.35127E-14	1.2124052430	1.28847E-05
0.08	1.13458484809	1.134584848086	5.41789E-14	1.1345712549	1.35932E-05
0.09	1.06350451192	1.063504511920	5.39568E-14	1.0634903953	1.41166E-05
0.1	0.99865792823	0.998657928235	5.37348E-14	0.9986434490	1.44792E-05

RIM = Rational interpolation method
DIRKM = Diagonally-Implicit Runge-Kutta method

IV. APPLICATION OF METHOD TO REAL LIFE PROBLEMS:

EXAMPLE 1:

The new cereal product is introduced through an advertising campaign to a population of 1 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 1 year, half of the population has heard of the product. How many will have heard of it by the end of 3 years?

MODELING THE ABOVE PROBLEM IN EXAMPLE 1:
Let y be the number (in millions) of people at time t who have heard of the product. This means that is the number of people who have not heard, and is the rate at which the population hears about the product. From the given assumption, you can write the differential equation as follows.

$$\frac{dy}{dt} = k(1-y) \quad (3.1)$$

Where $\frac{dy}{dt}$ is the rate of change of "y", $k =$ constant of proportionality and $(1-y)$ is the difference between y and 1. Normally by using separable variable method, and with the given assumptions, we obtain the value of $k = 0.6931471806 \approx 0.693$. So the differential equations becomes:

$$\frac{dy}{dt} = 0.693(1-y) \quad (3.2)$$

with theoretical solution as:

$$1-\exp(-0.693t) \quad (3.3)$$

XN (Yrs)	TSOL	YN	ERROR
1	0.49992640430	0.50007199770	1.455933978E-04
2	0.74992639889	0.75007199252	1.455936304E-04
3	0.87494479510	0.87505399050	1.091953981E-04
4	0.93746319403	0.93753599108	7.279704942E-05
5	0.96872699457	0.96877249280	4.549822955E-05
6	0.98436119573	0.98438849471	2.729898213E-05
7	0.99217944692	0.99219537135	1.592443225E-05
8	0.99608914790	0.99609824759	9.099690499E-06
9	0.99804428613	0.99804940471	5.118584338E-06
10	0.99902199913	0.99902484279	2.843662671E-06

To verify the solution given by the diagonally implicit method, we use the Exact solution (particular solution), which is $y(t) = 1-\exp(-0.693t)$, where "t" is time.

From $y(t) = 1-\exp(-0.693t)$, and $t = 3years$ we have:

$$y(3) = 0.87494 \approx 0.875 \quad (3.4)$$

Or 875,000 people would have heard of the product. This is correct, as can be seen in the table above from the result of the method at the end of the 3rd year.

EXAMPLE 2:
A tank contains 40 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing half water and half alcohol is added to the tank at the rate of 4 gallons per minute. At the same time, the tank is being drained at the rate of 4 gallons per minute. Assuming that the solution is stirred constantly, how much alcohol will be in the tank after 10 minutes?

MODELING THE ABOVE PROBLEM IN EXAMPLE 2:

Let y be the number of gallons of alcohol in the tank at any time t. The percent of alcohol in the 40-gallon tank at any time is $\frac{y}{40}$. Moreover, because 4 gallons of solution are being drained each minute, the rate of change of y is:

$$\frac{dy}{dx} = -4\left(\frac{y}{40}\right) + 2 \tag{3.5}$$

where 2 represents the number of gallons of alcohol entering each minute in the 50% solution. Here the exact solution to (3.5) is given as:

$$y(x) = 20 - 16\exp\left(-\frac{x}{10}\right) \tag{3.6}$$

where x is time: Implementing the diagonally implicit method, we obtain:

XN(minutes)	TSOL	YN	ERROR
1	5.52260131142	5.52260444167	3.130248615E-06
2	6.90030795075	6.90031361548	5.664731538E-06
3	8.14690846909	8.14691615758	7.688490756E-06
4	9.27487926343	9.27488853921	9.275777831E-06
5	10.29550944460	10.29551993594	1.049133745E-05
6	11.21901382250	11.21902521404	1.139154439E-05
7	12.05463513934	12.05464716475	1.202541025E-05
8	12.81073657412	12.81074900960	1.243547427E-05
9	13.49488544415	13.49489810274	1.265859136E-05
10	14.11392894126	14.11394166789	1.272662876E-05

Again verifying the result with x (time)=10 in (3.6), we have:

$$y(x) = 20 - 16\exp\left(-\frac{x}{10}\right)$$

$$y(10) = 20 - 16\exp\left(-\frac{10}{10}\right)$$

$$= 20 - 16\exp(-1)$$

$$= 14.11392894$$

$$\approx 14.114$$

This is also in line with the result from the method.

V. DISCUSSION:

From the tables above, it can be seen that the rational interpolation produced results that correspond with the exact solutions at each steps of iterations. Correspondingly, the errors at each step is at minimum compared to those produced by the other existing method. In a similar manner, implementing the same rational interpolation method on two real life problems in the above examples above, the results they produced, again, gave accurate solution as expected from the exact solutions. This is a sufficient prove of the validity of the rational interpolation method, and it can be used to proffer solution to both stiff, non-stiff and real life problems.

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